1.1 Division by 0

There are a number uncharitable comments one may make about division by zero. Here are a few.

1. Suppose there is a number x for which $x = 2 \div 0$. Then $0 \cdot x = 2$. But, for every number x, $0 \cdot x = 0$. So, there is some number that when multiplied by 0 equals both 0 and 2.

2. Division by a is defined as multiplication by the multiplicative inverse of a, written $\frac{1}{a}$ or a^{-1}. But 0 has no multiplicative inverse, since for all x, $0 \cdot x = 0$ and $0 \neq 1$. So division by zero is undefined.

3. This is possibly the most amusing comment. Suppose division by zero does result in a number. The product of a number and 0 is 0. Let a and b be any numbers. Then,

$$a \cdot 0 = b \cdot 0$$

dividing both sides by 0,

$$\frac{0 \cdot a}{0} = \frac{0 \cdot b}{0}$$

$$a = b.$$

And we have just shown that all numbers are the same number. One number, many names. But names all for the same number. Ah, numbers! We probably had too many of them anyway.

Would things be any better were we to suppose that "0"names an as yet undiscovered number, call it ζ?

No sense in beating a dead horse as they say. So we'll leave division by 0 alone now.

1.2 Being perplexed

The other day, we asked about $\lim_{x\to\infty}(x \sin \frac{1}{x})$. Writing $\lim_{x\to\infty}(x) \cdot \lim_{x\to\infty}(\sin \frac{1}{x}) = \infty \cdot 0$ was no help. We then proceeded slightly less naively.

$$\lim_{x\to\infty}\left(x \sin \frac{1}{x}\right) = \lim_{x\to\infty}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right)\left(x \sin \frac{1}{x}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\sin \frac{1}{x}}{\frac{1}{x}}\right)$$

$$= \frac{0}{0} \tag{1.1}$$

Not too illuminating, since undefined.

Next we tried substituting u for $\frac{1}{x}$.

$$u = \frac{1}{x},\ x = \frac{1}{u},\ x \to \infty \implies u \to 0.$$

Then

$$\begin{aligned} \lim_{x\to\infty} \left(x \sin \frac{1}{x} \right) &= \lim_{x\to 0} \left(\frac{1}{u} \right) (\sin u) \\ &= 1. \end{aligned} \tag{1.2}$$

What a difference a name makes!

Someone wondered whether equation (1.1) and equation (1.2) together imply that $\frac{0}{0} = 1$.

The answer is "No." The statement $\frac{0}{0} = 1$ is true just in case "$\frac{0}{0}$"and "1"are names fro the same number. But no number is named by "$\frac{0}{0}$."

Does equation (1.1) show that $\lim_{x\to\infty}(x \sin \frac{1}{x})$ is meaningless?

No. We simply rewrote the expression $\lim_{x\to\infty}(x \sin \frac{1}{x})$ in a way that concealed rather than revealed the number named by "$\lim_{x\to\infty}(x \sin \frac{1}{x})$".